

Percolation in the Sherrington-Kirkpatrick Spin Glass

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Abstract

We present extended versions and give detailed proofs of results concerning percolation (using various sets of two-replica bond occupation variables) in Sherrington-Kirkpatrick spin glasses (with zero external field) that were first given in an earlier paper by the same authors. We also explain how ultrametricity is manifested by the densities of large percolating clusters. Our main theorems concern the connection between these densities and the usual spin overlap distribution. Their corollaries are that the ordered spin glass phase is characterized by a unique percolating cluster of maximal density (normally coexisting with a second cluster of nonzero but lower density). The proofs involve comparison inequalities between SK multireplica bond occupation variables and the independent variables of standard Erdős-Rényi random graphs.

KEY WORDS: spin glass; percolation; Sherrington-Kirkpatrick model; Fortuin-Kasteleyn; random graphs

1 Introduction

In Ising ferromagnets (with no external field), it is well known that the ordered (broken symmetry) phase manifests itself within the associated Fortuin-Kasteleyn (FK) random cluster representation [1] by the occurrence of a single positive density percolating cluster (see [2]). In a recent paper [3], we investigated the nature of spin glass ordering within the FK and other graphical representations and concluded that the percolation signature of the spin glass phase is the presence of a single *two-replica* percolating network of *maximal* density, which typically coexists with a second percolating network of lower density. The evidence presented in that paper for this conclusion was two-fold: suggestive numerical results in the case of the three-dimensional Edwards-Anderson (EA) spin glass [4] and rigorous results for the Sherrington-Kirkpatrick (SK) spin glass [5].

In this paper, we expand on those results for the SK model in several ways. First, we give much more detailed proofs, both for two-replica FK (TRFK) percolation and for the different percolation of “blue” bonds in the two-replica graphical representation studied earlier by Chayes, Machta and Redner [6, 7] (CMR). Second, we go beyond the $\pm J$ SK model (as treated in [3]) to handle quite general choices of the underlying distribution ρ for the individual coupling variables, including the usual Gaussian case. Third, we organize the results (see in particular Theorems 1 and 2) in such a way as to separate out (see Theorems 4 and 5) those properties of the overlap distribution for the supercritical SK model that are needed to prove related properties about percolation structure. Such a separation is called for because many properties of the overlap distribution that are believed to be valid based on the Parisi ansatz (see [8]) for the SK model have not yet been rigorously proved.

Another way we expand on the results of the previous paper is to present (in Section 3) an analysis of the percolation signature of ultrametricity in the SK model, which is expected to occur, based on the Parisi analysis, but has not yet been proved rigorously. That is, we describe (see Theorems 6, 7 and 8) how the percolation cluster structure of multiple networks of differing densities in the context of three replicas would exhibit ultrametricity. We note that as a spinoff of Theorem 8, we have (at least in the SK model — numerical investigations for the EA model have not yet been done) a third graphical percolation signature of the spin glass transition beyond the two analyzed in our earlier work — namely one involving uniqueness of the maximum density percolating network (one out of *four* clusters) in a three replica mixed CMR-FK representation.

In addition to these extensions of our earlier results and as we note in a remark at the end of this introductory section, the technical machinery we develop in this paper can be used to obtain other results for the SK model, such as an analysis of large cluster densities at or near the critical

point. Before getting to that, we first give an outline of the other sections of the paper.

In Section 2 we describe the SK models and the CMR and TRFK percolation occupation variables we will be dealing with throughout. We then present our main results, starting with Theorems 1 and 2 which relate, in the limit $N \rightarrow \infty$, the densities of the largest percolation clusters to the overlap distribution for the CMR and TRFK representations respectively. Then, after stating a known basic result (Theorem 3) about the vanishing of the overlap in the subcritical (and critical) SK model, we present Theorems 4 and 5, which give respectively the CMR and TRFK percolation signatures of the SK phase transition, under various assumptions about the SK overlap distribution. Results relating percolation structure and ultrametricity in the SK model are presented in Section 3. Then in Section 4, we present all the proofs. That section begins with three lemmas that are the technical heart of the paper and explain why one can compare, via two-sided stochastic domination inequalities, SK percolation occupation variables to the independent variables of Erdős-Rényi random graphs [9]. A key feature of these comparison results is that they are done only *after conditioning on the values of the spin variables in all the replicas being considered*. This feature helps explain why the size of the overlap is crucial — because it determines the sizes of the various “sectors” of vertices (e.g., those where the spins in two replicas agree or those where they disagree) within which the comparisons can be made.

Remark 1.1 *The first part of Theorem 5 says that for $\beta \leq 1$, the size of the largest TRFK doubly occupied cluster is $o(N)$ or equivalently that its density $D_1^{TRFK} = o(1)$ as $N \rightarrow \infty$. But the proof (see the TRFK part of Lemma 4.2) combined with known results about $G(N, p_N)$, the Erdős-Rényi random graph with N vertices and independent edge occupation probability p_N , implies quite a bit more — that D_1^{TRFK} is $O(\log N/N)$ for $\beta < 1$ and $O(N^{2/3}/N)$ for $\beta = 1$. Even more is implied in the critical case. E.g., in the critical scaling window, where $\beta = \beta_N = 1 + \lambda/N^{1/3}$, the largest clusters, of size proportional to $N^{2/3}$, behave exactly like those occurring in a pair of independent random graphs (see, e.g., [10, 11, 12, 13]) — i.e., as $N \rightarrow \infty$, the limiting distribution of $(N^{1/3} D_1^{TRFK}, N^{1/3} D_2^{TRFK}, \dots)$ is the same as that obtained by taking two independent copies of $G(N/2, 2(\beta_N)^2/N)$, combining the sizes of the largest clusters in the two copies, then rank ordering them and dividing by $N^{2/3}$. One can also show that for $\beta > 1$, the size of the third largest TRFK cluster behaves like that of the second largest cluster in a single copy of the supercritical Erdős-Rényi random graph — i.e., $O(\log N)$ [9]. But that derivation requires a strengthened version of Lemma 4.2 and further arguments, which will not be presented in this paper.*

2 Main Results

Before stating the main results, we specify the random variables we will be dealing with. For specificity, we choose a specific probabilistic coupling so that even though we deal with two different graphical representations, and a range of inverse temperatures β , we define all our random variables for the system of positive integer size N on a single probability space. The corresponding probability measure will be denoted \mathbf{P}_N (with \mathbf{P} denoting probability more generically).

For each N , we have three types of random variables: real-valued couplings $\{J_{ij}\}_{1 \leq i < j \leq N}$, Ising ± 1 -valued spins $\{\sigma_i\}_{1 \leq i \leq N}$ and $\{\tau_i\}_{1 \leq i \leq N}$ for each of two replicas, and a variety of percolation $\{0, 1\}$ -valued bond occupation variables which we will define below. These random variables and their joint distributions depend on both N and β (although we have suppressed that in our notation), but to define them, we rely on other sets of real-valued random variables not depending on N or β : $\{K_{ij}\}_{1 \leq i < j < \infty}$ and $\{U_{ij}^\ell\}_{1 \leq i < j < \infty}$ for each replica indexed by $\ell = 1$ or 2 . (In later sections, we will consider more than two replicas.) Each of these sets is an i.i.d. family and the different sets are mutually independent. The U_{ij}^ℓ 's are independent mean one exponentials and will be used to define the bond occupation variables (conditionally on the couplings and spins). The K_{ij} 's, which determine the J_{ij} 's for given N (and β) by $J_{ij} = K_{ij}/\sqrt{N}$, have as their common distribution a probability measure ρ on the real line about which we make the following assumptions: ρ is even ($d\rho(x) = d\rho(-x)$) with no atom at the origin ($\rho(\{0\}) = 0$), variance one ($\int_{-\infty}^{+\infty} x^2 d\rho(x) = 1$) and a finite moment generating function ($\int_{-\infty}^{+\infty} e^{tx} d\rho(x) < \infty$ for all real t). The two most common choices are the Gaussian (where ρ is a mean zero, variance one normal distribution) and the $\pm J$ (where ρ is $(\delta_1 + \delta_{-1})/2$) spin glasses.

For a given N and β , we have already defined the couplings J_{ij} . The conditional distribution, given the couplings, of the spin variables σ, τ for the two replicas, is that of an independent sample from the Gibbs distribution; i.e.,

$$\text{const} \times \exp \left[\beta \sum_{1 \leq i < j \leq N} J_{ij} (\sigma_i \sigma_j + \tau_i \tau_j) \right]. \quad (1)$$

It remains to define the percolation bond occupation variables of interest, given the couplings and the spins. Two of these are the FK (random cluster) variables — one set for each replica; we will denote these \mathbf{n}_{ij}^ℓ for $\ell = 1$ (corresponding to the first (σ) replica) and $\ell = 2$ (corresponding to the second (τ) replica). These may be constructed as follows. For a given $i < j$, if the bond $\{i, j\}$ is unsatisfied in the first replica — i.e., if $J_{ij}\sigma_i\sigma_j < 0$, then set $\mathbf{n}_{ij}^1 = 0$; if the bond is satisfied, then set $\mathbf{n}_{ij}^1 = 1$ if $U_{ij}^1 \leq 2\beta|J_{ij}|$ (i.e., with probability $1 - \exp(-2\beta|J_{ij}|)$) and otherwise set it to zero. Define \mathbf{n}_{ij}^2 similarly using the second (τ) replica. We will be particularly interested in the

percolation properties of the variables $\mathbf{n}_{ij} = \mathbf{n}_{ij}^1 \mathbf{n}_{ij}^2$ that describe *doubly FK-occupied* bonds. We will use the acronym TRFK (for Two Replica FK) to denote various quantities built out of these variables.

There is another two-replica graphical representation, introduced by Chayes, Machta and Redner [6, 7] (which we will denote by CMR) that we will also consider. This representation in general is described in terms of three types of bonds which may be thought of as those that are colored blue or red or else are uncolored. One way of defining the blue bonds, whose occupation variables we will denote by \mathbf{b}_{ij} , is that $\mathbf{b}_{ij} = 1$ if $\{i, j\}$ is satisfied in *both* replicas *and* also either $\mathbf{n}_{ij}^1 = 1$ or $\mathbf{n}_{ij}^2 = 1$ or both (which occurs with probability $1 - \exp(-4\beta|J_{ij}|)$); otherwise $\mathbf{b}_{ij} = 0$. We will be interested in the percolation properties of the blue bonds. Although we will not be using them in this paper, we note that a bond $\{i, j\}$ is colored red if and only if $\sigma_i \sigma_j \tau_i \tau_j = -1$ (or equivalently $\{i, j\}$ is satisfied in exactly one of the two replicas) *and* also that satisfied bond is FK-occupied (which occurs with probability $1 - \exp(-2\beta|J_{ij}|)$).

A key role in the theory of spin glasses, and this will also be the case for their percolation properties, is played by the Parisi (spin) overlap. For a given N and β , this overlap is the random variable,

$$Q = Q(N, \beta) = N^{-1} \sum_{1 \leq i \leq N} \sigma_i \tau_i. \quad (2)$$

Closely related to the overlap are the densities (i.e., the fractions of sites out of N) $D_a = D_a(N, \beta)$ and $D_d = D_d(N, \beta)$ of the collections of sites where the spins of the two replicas respectively agree and disagree with each other. Since $Q = D_a - D_d$ and $D_a + D_d = 1$, one can express $D_{\max} = \max\{D_a, D_d\}$ and $D_{\min} = \min\{D_a, D_d\}$ as $D_{\max} = [1 + |Q|]/2$ and $D_{\min} = [1 - |Q|]/2$.

It should be clear from our definitions of the various bond occupation variables that if one of i, j is in the collection of agree sites and the other is in the collection of disagree sites, then the bond $\{i, j\}$ is satisfied in exactly one of the two replicas and so $\{i, j\}$ can neither be a TRFK occupied bond nor a CMR blue bond. So percolation (i.e., occurrence of giant clusters containing order N of the sites) can only occur separately within the agree or within the disagree collections of sites. Our results concern when this happens and its connection with the spin glass phase transition in the SK model via the overlap random variable Q .

We will first state two general theorems relating the occurrence of giant clusters to the behavior of Q and then state a number of corollaries. The corollaries depend for their applicability on results about the nature of Q in the SK model, some of which have been and some of which have not yet been derived rigorously. The first theorem concerns CMR percolation. We denote the density of the k 'th largest CMR blue cluster by $D_k^{\text{CMR}}(N, \beta)$.

Theorem 1 *In the CMR representation, for any $0 < \beta < \infty$, the following three sequences of random variables tend to zero in probability, i.e., the \mathbf{P}_N -probability that the absolute value of the random variable is greater than ε tends to zero as $N \rightarrow \infty$ for any $\varepsilon > 0$.*

$$D_1^{CMR}(N, \beta) - \left[\frac{1 + |Q(N, \beta)|}{2} \right] \rightarrow 0. \quad (3)$$

$$D_2^{CMR}(N, \beta) - \left[\frac{1 - |Q(N, \beta)|}{2} \right] \rightarrow 0. \quad (4)$$

$$D_3^{CMR}(N, \beta) \rightarrow 0. \quad (5)$$

To state the next theorem, we define $\theta(c)$ for $c \in [0, \infty)$ to be the order parameter for mean-field percolation — i.e., the asymptotic density (fraction of sites out of N as $N \rightarrow \infty$) of the largest cluster in $G(N, c/N)$, the Erdős-Rényi random graph with occupation probability c/N independently for each edge in the complete graph of N sites [9]. It is a standard fact [9] that $\theta(c)$ is zero for $0 \leq c \leq 1$ and for $c > 1$ is the strictly positive solution of

$$\theta = 1 - e^{-c\theta}. \quad (6)$$

Theorem 2 *In the TRFK representation, for any $0 < \beta < \infty$, the following limits (in probability) are valid as $N \rightarrow \infty$ for D_k^{TRFK} , the density of the k 'th largest TRFK doubly occupied cluster:*

$$D_1^{TRFK}(N, \beta) - \theta(2\beta^2 \frac{1 + |Q(N, \beta)|}{2}) \left[\frac{1 + |Q(N, \beta)|}{2} \right] \rightarrow 0. \quad (7)$$

$$D_2^{TRFK}(N, \beta) - \theta(2\beta^2 \frac{1 - |Q(N, \beta)|}{2}) \left[\frac{1 - |Q(N, \beta)|}{2} \right] \rightarrow 0. \quad (8)$$

$$D_3^{TRFK}(N, \beta) \rightarrow 0. \quad (9)$$

We now denote by $P(N, \beta)$ the probability distribution of the overlap $Q(N, \beta)$. This is the Parisi overlap distribution, averaged over the disorder variables $\mathcal{K} = \{K_{ij}\}_{1 \leq i \leq j < \infty}$. The unaveraged overlap distribution requires conditioning on \mathcal{K} . So, for example, we have, for $q \in [-1, 1]$,

$$P(N, \beta)([-1, q]) = \text{Av}[\mathbf{P}_N(Q(N, \beta) \leq q | \mathcal{K})], \quad (10)$$

where Av denotes the average over the disorder distribution of \mathcal{K} .

The quantity $E(Q(N, \beta)^2 | \mathcal{K})$ is closely related to $\sum_{1 \leq i < j \leq N} [E(\sigma_i \sigma_j | \mathcal{K})]^2$ (see, e.g., [14], Lemma 2.2.10) which in turn is closely related to the derivative of the finite volume free energy (see, e.g., [15], Prop. 4.1). It then follows that $E(Q(N, \beta)^2) \rightarrow 0$ as $N \rightarrow \infty$ first for $\beta < 1$ ([15], Prop. 2.1) and then (using results of [16, 17, 18] and of [19, 20] — see also [21]) also for $\beta = 1$. This implies the following theorem, one of the basic facts in the mathematical theory of the SK model.

Theorem 3 For $\beta \leq 1$, $Q(N, \beta) \rightarrow 0$ (in probability) as $N \rightarrow \infty$, or equivalently $P(N, \beta) \rightarrow \delta_0$.

The situation regarding rigorous results about the nonvanishing of $Q(N, \beta)$ as $N \rightarrow \infty$ for $\beta > 1$ is less clean. For example, using results from the references cited just before Theorem 3, it follows that $E(Q(N, \beta)^2)$ has a limit as $N \rightarrow \infty$ for all β , related by a simple identity to the derivative at that β of the infinite-volume free energy, $\mathcal{P}(\beta)$, given by the Parisi variational formula, $\mathcal{P}(\beta) = \inf_m \mathcal{P}(m, \beta)$, where the inf is over distribution functions $m(q)$ with $q \in [0, 1]$ — see [19]. Furthermore, since for $\beta > 1$, $\mathcal{P}(\beta)$ is strictly below the “annealed” free energy [22] (which equals $\mathcal{P}(\delta_0, \beta)$, where δ_0 is the distribution function for the unit point mass at $q = 0$), it follows (see [21]) from Lipschitz continuity of $\mathcal{P}(m, \beta)$ in m [17, 20] that

$$\lim_{N \rightarrow \infty} E(Q(N, \beta)^2) > 0 \text{ for all } \beta > 1. \quad (11)$$

However, it seems that it is not yet proved in general that $Q(N, \beta)$ has a unique limit (in distribution) nor very much about the precise nature of any limit. In order to explain the corollaries of our main theorems without getting bogged down in these unresolved questions about the SK model, we will list various properties which are expected to be valid for (at least some values of) $\beta > 1$ and then use those as assumptions in our corollaries. Some related comments are given in Remark 2.2 below.

Possible Behaviors of the Supercritical Overlap. For $\beta > 1$, $P(N, \beta)$ converges as $N \rightarrow \infty$ to some P_β with the following properties.

- Property P1: $P_\beta(\{0\}) = 0$.
- Property P2: $P_\beta(\{-1, +1\}) = 0$.
- Property P3: $P_\beta([-1, -1 + (1/\beta^2)] \cup [1 - (1/\beta^2), 1]) = 0$.

If one defines $q_{EA}(\beta)$, the Edwards-Anderson order parameter (for the SK model), to be the supremum of the support of P_β , and one assumes that P_β has point masses at $\pm q_{EA}(\beta)$, then Properties P2 and P3 reduce respectively to $q_{EA}(\beta) < 1$ and $q_{EA}(\beta) < 1 - (1/\beta^2)$. Weaker versions of the three properties that do not require existence of a limit for $P(N, \beta)$ as $N \rightarrow \infty$ are as follows.

- Property P1':

$$\lim_{\varepsilon \downarrow 0} \limsup_{N \rightarrow \infty} \mathbf{P}_N(|Q(N, \beta)| < \varepsilon) = 0. \quad (12)$$

- Property P2':

$$\lim_{\varepsilon \downarrow 0} \limsup_{N \rightarrow \infty} \mathbf{P}_N(|Q(N, \beta)| > 1 - \varepsilon) = 0. \quad (13)$$

- Property P3'

$$\lim_{\varepsilon \downarrow 0} \limsup_{N \rightarrow \infty} \mathbf{P}_N(|Q(N, \beta)| > 1 - (1/\beta^2) - \varepsilon) = 0. \quad (14)$$

We will state the next two theorems in a somewhat informal manner and then provide a more precise meaning in Remark 2.1 below.

Theorem 4 (Corollary to Theorem 1) *In the CMR representation, for any $0 < \beta \leq 1$, there are exactly two giant blue clusters, each of (asymptotic) density $1/2$. For $1 < \beta < \infty$, there are either one or two giant blue clusters, whose densities add to 1; there is a unique one of (maximum) density in $(1/2, 1]$ providing Property P1 (or P1') is valid and there is another one of smaller density in $(0, 1/2)$ providing Property P2 (or P2') is valid.*

Theorem 5 (Corollary to Theorem 2) *In the TRFK representation, there are no giant doubly occupied clusters for $\beta \leq 1$. For $1 < \beta < \infty$, there are either one or two giant doubly occupied clusters with a unique one of maximum density providing Property P1 (or P1') is valid and another one of smaller (but nonzero) density providing Property P3 (or P3') is valid.*

Remark 2.1 *For $0 < \beta \leq 1$, Theorem 4 states that $(D_1^{\text{CMR}}(N, \beta), D_2^{\text{CMR}}(N, \beta), D_3^{\text{CMR}}(N, \beta))$ converges (in probability or equivalently in distribution) to $(1/2, 1/2, 0)$ while Theorem 5 states that the corresponding triple of largest TRFK cluster densities converges to $(0, 0, 0)$. A precise statement of the results for $\beta \in (1, \infty)$ is a bit messier because it has not been proved that there is a single limit in distribution of these cluster densities, although since the densities are all bounded (in $[0, 1]$) random variables, there is compactness with limits along subsequences of N 's. For example, in the CMR case, assuming Properties P1' and P2', the precise statement is that any limit in distribution of the triplet of densities is supported on $\{(1/2 + a, 1/2 - a, 0) : a \in (0, 1/2)\}$. Precise statements for the other cases treated in the two theorems are analogous.*

Remark 2.2 *Although Property P1' does not seem to have yet been rigorously proved (for any $\beta > 1$), a weaker property does follow from (11). Namely, that for all $\beta > 1$, the limit in (12) is strictly less than one. Weakened versions of portions of Theorems 4 and 5 for $\beta > 1$ follow — e.g., any limit in distribution of the triplet of densities in Remark 2.1 must assign strictly positive probability to $\{(1/2 + a, 1/2 - a, 0) : a \in (0, 1/2)\}$.*

3 Ultrametricity and Percolation

In this section, in order to discuss ultrametricity, which is expected to occur in the supercritical SK model (see [8]), we consider three replicas, whose spin variables are denoted $\{\sigma_i^\ell\}$ for $\ell =$

1, 2, 3. We denote by \mathbf{n}_{ij}^ℓ the FK occupation variables for replica ℓ and by $\mathbf{b}_{ij}^{\ell m}$ the CMR blue bond occupation variables for the pair of replicas ℓ, m . Thus \mathbf{b}_{ij}^{12} corresponds in our previous notation to \mathbf{b}_{ij} . We also denote by $Q^{\ell m} = Q^{\ell m}(N, \beta)$ the overlap defined in (2), but with σ, τ replaced by σ^ℓ, σ^m .

Let us denote by $P^3(N, \beta)$ the distribution of the triple of overlaps (Q^{12}, Q^{13}, Q^{23}) . Ultrametricity concerns the nature of the limits as $N \rightarrow \infty$ of $P^3(N, \beta)$, as follows, where we define

$$\mathbf{R}_{\text{ultra}}^3 = \{(x, y, z) : |x| = |y| \leq |z| \text{ or } |x| = |z| \leq |y| \text{ or } |y| = |z| \leq |x|\}. \quad (15)$$

Possible Ultrametric Behaviors of the Supercritical Overlap. For $\beta > 1$, $P^3(N, \beta)$ converges to some P_β^3 as $N \rightarrow \infty$ with

- Property P4: $P_\beta^3(\mathbf{R}_{\text{ultra}}^3) = 1$.

We will generally replace this property by a weakened version, P4', in which it is not assumed that there is a single limit P_β^3 as $N \rightarrow \infty$ but rather the same property is assumed for every subsequence limit. There is another property that simplifies various of our statements about how ultrametricity is manifested in the sizes of various percolation clusters. This property, which, like ultrametricity, is expected to be valid in the supercritical SK model (see [23], where this property is discussed and also numerically tested in the three-dimensional EA model) is the following.

- Property P5: $P_\beta^3(\{(x, y, z) : xyz \geq 0\}) = 1$.

Again we will use a weaker version P5' in which it is not assumed that there is a single limit P_β^3 as $N \rightarrow \infty$.

One formulation of ultrametricity using percolation clusters is the next theorem, an immediate corollary of Theorem 1, in which we denote by $D_j^{\ell m} = D_j^{\ell m}(N, \beta)$ the density of sites in $\mathcal{C}_j^{\ell m}$, the j 'th largest cluster formed by the bonds $\{i, j\}$ with $\mathbf{b}_{ij}^{\ell m} = 1$ (i.e., the j 'th largest CMR blue cluster for the pair $\{\ell, m\}$ of replicas). Note that D_j^{12} coincides in our previous notation with D_j^{CMR} .

Theorem 6 (Corollary to Theorem 1) *For $1 < \beta < \infty$, assuming Property P4', any subsequence limit in distribution as $N \rightarrow \infty$ of the triple $(D_1^{12} - D_2^{12}, D_1^{13} - D_2^{13}, D_1^{23} - D_2^{23})$ is supported on $\mathbf{R}_{\text{ultra}}^3$.*

In our next two theorems, instead of looking at *differences* of densities, we express ultrametricity directly in terms of densities themselves. This is perhaps more interesting because rather than having three density differences, there will be *four* densities. We begin in Theorem 7 with a fully

CMR point of view with four natural non-empty *intersections* of CMR blue clusters. Then Theorem 8 mixes CMR and FK occupation variables to yield (four) other natural clusters.

There are a number of ways in which the four sets of sites in our fully CMR perspective can be defined, which turn out to be equivalent (for large N). One definition is as follows. For α, α' each taken to be either the letter a (for agree) or the letter d (for disagree), define $\Lambda_{\alpha\alpha'}(N, \beta)$ to be the set of sites $i \in \{1, \dots, N\}$ where σ_i^1 agrees (for $\alpha = a$) or disagrees (for $\alpha = d$) with σ_i^2 and σ_i^1 agrees (for $\alpha' = a$) or disagrees (for $\alpha' = d$) with σ_i^3 ; also denote by $D_{\alpha\alpha'}(N, \beta)$ the density of sites (i.e., the fraction of N) in $\Lambda_{\alpha\alpha'}(N, \beta)$. Then denote by $\mathcal{C}_{\alpha\alpha'}^{\ell m}$ the largest cluster (thought of as the collection of its sites) formed within $\Lambda_{\alpha\alpha'}$ by the $\mathbf{b}^{\ell m} = 1$ blue bonds. Finally, define $\mathcal{C}_{\alpha\alpha'} = \mathcal{C}_{\alpha\alpha'}^{12} \cap \mathcal{C}_{\alpha\alpha'}^{13}$, $D_{\alpha\alpha'}^{\text{CMR}}$ to be the density of sites (fraction of N) in $\mathcal{C}_{\alpha\alpha'}$ and $\hat{D}^{\text{CMR}}(N, \beta)$ to be the vector of four densities $(D_{aa}^{\text{CMR}}, D_{ad}^{\text{CMR}}, D_{da}^{\text{CMR}}, D_{dd}^{\text{CMR}})$. To state the next theorem, let $\mathbf{R}^{4,+}$ denote $\{(x_1, x_2, x_3, x_4) : \text{each } x_i \geq 0\}$ and define

$$\mathbf{R}_{\text{ultra}}^{4,+} = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^{4,+} : x_{(1)} > x_{(2)} \geq x_{(3)} = x_{(4)}\}, \quad (16)$$

where $x_{(1)}, x_{(2)}, x_{(3)}, x_{(4)}$ are the rank ordered values of x_1, x_2, x_3, x_4 .

Theorem 7 For $0 < \beta \leq 1$, $\hat{D}^{\text{CMR}}(N, \beta) \rightarrow (1/4, 1/4, 1/4, 1/4)$ (in probability) as $N \rightarrow \infty$. For $1 < \beta < \infty$ and assuming Properties $P1', P2', P4', P5'$, any limit in distribution of $\hat{D}^{\text{CMR}}(N, \beta)$ is supported on $\mathbf{R}_{\text{ultra}}^{4,+}$.

Remark 3.1 The equilateral triangle case where $|Q^{12}| = |Q^{13}| = |Q^{23}| = q_u$ corresponds to $x_{(1)} = (1 + 3q_u)/4$ and $x_{(2)} = x_{(3)} = x_{(4)} = (1 - q_u)/4$. The alternative isosceles triangle case with, say, $|Q^{12}| = q_u > |Q^{13}| = |Q^{23}| = q_\ell$ corresponds to $x_{(1)} = (1 + q_u + 2q_\ell)/4$, $x_{(2)} = (1 + q_u - 2q_\ell)/4$ and $x_{(3)} = x_{(4)} = (1 - q_u)/4$.

In the next theorem, we consider $\mathcal{C}_{\alpha\alpha'}^*$ defined as the largest cluster within $\Lambda_{\alpha\alpha'}(N, \beta)$ formed by bonds $\{i, j\}$ with $\mathbf{b}_{ij}^{12} \mathbf{n}_{ij}^3 = 1$ — i.e., bonds that are simultaneously CMR blue for the first two replicas and FK-occupied for the third replica. Let $\hat{D}^*(N, \beta)$ denote the corresponding vector of four densities. As in the previous theorem, we note that there are alternative, but equivalent for large N , definitions of the clusters and densities (e.g., as the four largest clusters formed by bonds $\{i, j\}$ with $\mathbf{b}_{ij}^{12} \mathbf{n}_{ij}^3 = 1$ in all of $\{1, \dots, N\}$ without a priori restriction to $\Lambda_{\alpha\alpha'}(N, \beta)$).

Theorem 8 For $0 < \beta \leq 1$, $\hat{D}^*(N, \beta) \rightarrow (0, 0, 0, 0)$ (in probability) as $N \rightarrow \infty$. For $1 < \beta < \infty$ and assuming Properties $P1', P3', P4', P5'$, any limit in distribution of $\hat{D}^*(N, \beta)$ is supported on $\mathbf{R}_{\text{ultra}}^{4,+}$.

Remark 3.2 Here the limiting densities are of the form $x_{(j)} = \theta(4\beta^2 d_{(j)}) d_{(j)}$ where $d_{(1)} = (1 + q_u + 2q_\ell)/4$, $d_{(2)} = (1 + q_u - 2q_\ell)/4$ and $d_{(3)} = d_{(4)} = (1 - q_u)/4$, with $q_\ell = q_u$ for the equilateral triangle case.

4 Proofs

Before giving the proofs of our main results, we present several key lemmas which are the technical heart of our proofs. We will use the notation \gg and \ll to denote stochastic domination (in the FKG sense) for either families of random variables or their distributions. E.g., for the m -tuples $X = (X_1, \dots, X_m)$ and $Y = (Y_1, \dots, Y_m)$ we write $X \ll Y$ or $Y \gg X$ to mean that $\mathbf{E}(h(X_1, \dots, X_m)) \leq \mathbf{E}(h(Y_1, \dots, Y_m))$ for every coordinatewise increasing function h (for which the two expectations exist).

Our key lemmas concern stochastic domination inequalities in the k -replica setting comparing conditional distributions of the couplings $\{K_{ij}\}$ or related bond occupation variables, when the spins $\sigma^1, \dots, \sigma^k$ are fixed, to product measures. These allow us to approximate percolation variables in the SK model by the *independent* variables of Erdős-Rényi random graphs when $N \rightarrow \infty$. Given probability measures ν_{ij} on \mathbf{R} for $1 \leq i < j \leq N$, we denote by $\text{Prod}_N(\{\nu_{ij}\})$ the corresponding product measure on $\mathbf{R}^{N(N-1)/2}$. We also will denote by $\rho[\gamma]$ the probability measure defined by $d\rho[\gamma](x) = e^{\gamma x} d\rho(x) / \int_{-\infty}^{+\infty} e^{\gamma x'} d\rho(x')$.

Lemma 4.1 Fix N, β, k and let $\tilde{\mu}_{N,\beta}^k$ denote the conditional distribution, given the spins $\sigma^1 = (\sigma_i^1 : 1 \leq i \leq N), \dots, \sigma^k$, of $\{K'_{ij} \equiv \varepsilon_{i,j} K_{ij}\}_{1 \leq i < j \leq N}$ (where the $\varepsilon_{i,j}$'s are any given ± 1 values). Then

$$\text{Prod}_N(\{\rho[\gamma_{ij}^{k,-}]\}) \ll \tilde{\mu}_{N,\beta}^k \ll \text{Prod}_N(\{\rho[\gamma_{ij}^{k,+}]\}), \quad (17)$$

where

$$\gamma_{ij}^{k,\pm} = \frac{\beta}{\sqrt{N}} [\varepsilon_{ij} (\sigma_i^1 \sigma_j^1 + \dots + \sigma_i^k \sigma_j^k) \pm k]. \quad (18)$$

Proof. Define the partition function

$$Z_{N,\beta} = Z_{N,\beta}(\{K'_{ij}\}) = \sum_{\sigma} \exp\left(\frac{\beta}{\sqrt{N}} \sum_{1 \leq i < j \leq N} (\varepsilon_{ij} \sigma_i \sigma_j) K'_{ij}\right), \quad (19)$$

where \sum_{σ} denotes the sum over all 2^N choices of $\sigma_i = \pm 1$ for $1 \leq i \leq N$. Thus the normalization constant in Equation (1) for two replicas is $(Z_{N,\beta})^{-2}$ and the k -replica marginal distribution for

$\{K'_{ij}\}$ is as follows, where $\text{Prod}_N(\{\rho\})$ denotes $\text{Prod}_N(\{\nu_{ij}\})$ with $\nu_{ij} \equiv \rho$, $\phi_{ij}^k = \varepsilon_{ij}(\sigma_i^1 \sigma_j^1 + \dots + \sigma_i^k \sigma_j^k)$ and $C_{N,\beta}$ is a normalization constant depending on the ϕ_{ij}^k 's but not on the K'_{ij} 's.

$$d\tilde{\mu}_{N,\beta}^k = C_{N,\beta} (Z_{N,\beta}(\{K'_{ij}\}))^{-k} \exp\left(\frac{\beta}{\sqrt{N}} \sum_{1 \leq i < j \leq N} \phi_{ij}^k K'_{ij}\right) \text{Prod}_N(\{\rho\}). \quad (20)$$

It is a standard fact about stochastic domination that if $\tilde{\mu}$ is a product probability measure on \mathbf{R}^m and g is an increasing — i.e., coordinatewise nondecreasing — (respectively, decreasing) function on \mathbf{R}^m (with $\int g d\tilde{\mu} < \infty$), then the probability measure $\tilde{\mu}_g$ defined as $d\tilde{\mu}_g(x_1, \dots, x_m) = g(x_1, \dots, x_m) d\tilde{\mu} / \int g d\tilde{\mu}$ satisfies $\tilde{\mu}_g \gg \tilde{\mu}$ (respectively, $\tilde{\mu}_g \ll \tilde{\mu}$). This follows from the fact that product measures satisfy the FKG inequalities — i.e., for f and g increasing $\int f g d\tilde{\mu} \geq \int f d\tilde{\mu} \int g d\tilde{\mu}$.

On the other hand, since each $\varepsilon_{ij} \sigma_i \sigma_j = \pm 1$, it follows from (19) that

$$Z_{N,\beta}^\pm \equiv Z_{N,\beta} e^{\pm(\beta/\sqrt{N}) \sum K'_{ij}} = \sum_{\sigma} e^{(\beta/\sqrt{N}) \sum \psi_{ij}^\pm(\sigma) K'_{ij}} \quad (21)$$

with each $\psi_{ij}^\pm = 0$ or ± 2 and hence each $\psi_{ij}^+ \geq 0$ (respectively, each $\psi_{ij}^- \leq 0$). Thus, as a function of the K'_{ij} 's, $Z_{N,\beta}^+$ is increasing and $(Z_{N,\beta}^+)^{-k}$ is decreasing while $Z_{N,\beta}^-$ is decreasing and $(Z_{N,\beta}^-)^{-k}$ is increasing. Combining this with the previous discussion about stochastic domination and product measures, we see that

$$C_{N,\beta}^- e^{-(\beta/\sqrt{N}) \sum (-kK'_{ij} + \phi_{ij}^k K'_{ij})} \text{Prod}_N(\{\rho\}) \ll \tilde{\mu}_{N,\beta}^k \ll C_{N,\beta}^+ e^{-(\beta/\sqrt{N}) \sum (+kK'_{ij} + \phi_{ij}^k K'_{ij})} \text{Prod}_N(\{\rho\}), \quad (22)$$

which is just Equation (17) since the $C_{N,\beta}^\pm$ are normalization constants. This completes the proof of Lemma 4.1.

The next two lemmas give stochastic domination inequalities for three different sets of occupation variables in terms of independent Bernoulli percolation variables. The first lemma covers the cases of CMR and TRFK variables involving two replicas and the second lemma deals with mixed CMR-FK variables involving three replicas. The parameters of the independent Bernoulli variables used for upper and lower bounds are denoted $p_{N,\beta}^{*,\natural}$, where $*$ denotes CMR or TRFK or 3 (for the mixed CMR-FK case) and \natural denotes u (for upper bound) or ℓ (for lower bound) and are as follows.

$$p_{N,\beta}^{*,u} = \int_0^\infty g_{N,\beta}^*(x) d\rho_{N,\beta}^*(x), \quad (23)$$

$$p_{N,\beta}^{*,\ell} = \int_0^\infty g_{N,\beta}^*(x) d\rho(x), \quad (24)$$

where

$$g_{N,\beta}^{\text{CMR}}(x) = 1 - e^{-4(\beta/\sqrt{N})x}, \quad (25)$$

$$g_{N,\beta}^{\text{TRFK}}(x) = (1 - e^{-2(\beta/\sqrt{N})x})^2, \quad (26)$$

$$g_{N,\beta}^3(x) = (1 - e^{-4(\beta/\sqrt{N})x})(1 - e^{-2(\beta/\sqrt{N})x}), \quad (27)$$

and

$$\rho_{N,\beta}^{\text{CMR}} = \rho_{N,\beta}^{\text{TRFK}} = \rho[4\beta/\sqrt{N}], \quad (28)$$

$$\rho_{N,\beta}^3 = \rho[6\beta/\sqrt{N}]. \quad (29)$$

Lemma 4.2 Fix $N, \beta, \sigma^1, \sigma^2$ and consider the conditional distributions $\hat{\mu}_{N,\beta}^{\text{CMR}}$ of $\{\mathbf{b}_{i,j} \equiv \mathbf{b}_{i,j}^{12}\}_{1 \leq i < j \leq N}$ and $\hat{\mu}_{N,\beta}^{\text{TRFK}}$ of $\{\mathbf{n}_{i,j} \equiv \mathbf{n}_{i,j}^1 \mathbf{n}_{i,j}^2\}_{1 \leq i < j \leq N}$. Then

$$\text{Prod}_N(p_{i,j}^{*,\ell} \delta_1 + (1 - p_{i,j}^{*,\ell}) \delta_0) \ll \hat{\mu}_{N,\beta}^* \ll \text{Prod}_N(p_{i,j}^{*,u} \delta_1 + (1 - p_{i,j}^{*,u}) \delta_0), \quad (30)$$

where $p_{i,j}^{*,\mathfrak{h}} = p_{N,\beta}^{*,\mathfrak{h}}$ if i, j are either both in Λ_a or both in Λ_d and $p_{i,j}^{*,\mathfrak{h}} = 0$ otherwise. The asymptotic behavior as $N \rightarrow \infty$ of the parameters appearing in these inequalities is

$$p_{N,\beta}^{\text{CMR},\mathfrak{h}} = \frac{2\beta}{\sqrt{N}} \left(\int_{-\infty}^{+\infty} |x| d\rho(x) \right) (1 + O(\frac{1}{\sqrt{N}})), \quad (31)$$

$$p_{N,\beta}^{\text{TRFK},\mathfrak{h}} = \frac{2\beta^2}{N} (1 + O(\frac{1}{\sqrt{N}})). \quad (32)$$

Proof. We begin with some considerations for the general case of k replicas and arbitrary ε_{ij} before specializing to what is used in this lemma. The case $k = 3$ will be used for the next lemma. Let $(\tilde{K}, \tilde{U})_N^k \equiv (\tilde{K}_{ij}, \tilde{U}_{ij}^1, \dots, \tilde{U}_{ij}^k)_N$ denote random variables (with $1 \leq i < j \leq N$) whose joint distribution is the conditional distribution of $(K'_{ij}, U_{ij}^1, \dots, U_{ij}^k)_N$ given $\sigma^1, \dots, \sigma^k$. Note that the \tilde{U}_{ij}^ℓ 's are independent of the \tilde{K}_{ij} 's and (like the U_{ij}^ℓ 's) are independent mean one exponential random variables. Let $(K^{\text{ind}}[\{\gamma_{ij}\}], U^{\text{ind}})_N^k \equiv (K_{ij}^{\text{ind}}[\gamma_{ij}], U_{ij}^{\text{ind},1}, \dots, U_{ij}^{\text{ind},k})_N$ denote mutually independent random variables with $K_{ij}^{\text{ind}}[\gamma_{ij}]$ distributed by $\rho[\gamma_{ij}]$ and each $U_{ij}^{\text{ind},\ell}$ a mean one exponential random variable.

Lemma 4.1 says that

$$(K^{\text{ind}}[\{\gamma_{ij}^{k,-}\}])_N^k \ll (\tilde{K})_N^k \ll (K^{\text{ind}}[\{\gamma_{ij}^{k,+}\}])_N^k \quad (33)$$

and it immediately follows that

$$(K^{\text{ind}}[\{\gamma_{ij}^{k,-}\}], -U^{\text{ind}})_N^k \ll (\tilde{K}, -\tilde{U})_N^k \ll (K^{\text{ind}}[\{\gamma_{ij}^{k,+}\}], -U^{\text{ind}})_N^k. \quad (34)$$

We now take $k = 2$, choose $\varepsilon_{ij} = \sigma_i^1 \sigma_j^1$ and note that $\mathbf{n}_{ij} \equiv \mathbf{n}_{ij}^1 \mathbf{n}_{ij}^2 = 0 = \mathbf{b}_{ij}$ unless i, j are either both in Λ_a or both in Λ_d , in which case we have $\sigma_i^1 \sigma_j^1 = \sigma_i^2 \sigma_j^2$, $\gamma_{ij}^{2,+} = 4\beta/\sqrt{N}$ and

$\gamma_{ij}^{2,-} = 0$. Furthermore \mathbf{n}_{ij}^ℓ is the indicator of the event that $2\beta K'_{ij} + (-U_{ij}^\ell) \geq 0$ while $\mathbf{b}_{ij} \equiv \mathbf{b}_{ij}^{12}$ is the indicator of the event that $\mathbf{n}_{ij}^1 + \mathbf{n}_{ij}^2 \geq 1$ and $K'_{ij} \geq 0$, so that these occupation variables (for such i, j) are increasing functions of $(K', -U)_N^2$. Let us now define $\tilde{\mathbf{n}}_{ij}$, $\tilde{\mathbf{b}}_{ij}$ and $\mathbf{n}_{ij}^{\text{ind},\pm}$, $\mathbf{b}_{ij}^{\text{ind},\pm}$ as the same increasing functions, respectively, of $(\tilde{K}, -\tilde{U})_N^2$ and $(K^{\text{ind}}[\{\gamma_{ij}^{k,\pm}\}], -U^{\text{ind}})_N^2$ providing i, j are either both in Λ_a or both in Λ_d , and otherwise set these occupation variables to zero. Then, as a consequence of (34), we have

$$(\mathbf{n}^{\text{ind},-})_N^2 \ll (\tilde{\mathbf{n}})_N^2 \ll (\mathbf{n}^{\text{ind},+})_N^2, \quad (35)$$

and

$$(\mathbf{b}^{\text{ind},-})_N^2 \ll (\tilde{\mathbf{b}})_N^2 \ll (\mathbf{b}^{\text{ind},+})_N^2. \quad (36)$$

To complete the proof of Lemma 4.2, it remains to obtain the claimed formulas for the various Bernoulli occupation parameters. E.g., for the case $* = \text{TRFK}$ and $\mathfrak{h} = u$, we have for i, j either both in Λ_a or both in Λ_d ,

$$\begin{aligned} p_{N,\beta}^{\text{TRFK},u} &= \mathbf{P}(\mathbf{n}_{ij}^{\text{ind},+} = 1) \\ &= \mathbf{P}(2\beta K_{ij}^{\text{ind}} \gamma_{ij}^{2,+} / \sqrt{N} \geq U_{ij}^{\text{ind},1}, U_{ij}^{\text{ind},2}) \\ &= \int_0^\infty (1 - e^{-2(\beta/\sqrt{N})x})^2 d\rho[4\beta/\sqrt{N}](x), \end{aligned} \quad (37)$$

as given by Equations (23), (26) and (28). We leave the checking of the other three cases (for $k = 2$) and the straightforward derivation of the asymptotic behavior as $N \rightarrow \infty$ for all the parameters to the reader. This completes the proof of Lemma 4.2.

Lemma 4.3 *Fix $N, \beta, \sigma^1, \sigma^2, \sigma^3$ and consider the conditional distribution $\hat{\mu}_{N,\beta}^3$ of $\{\mathbf{b}_{i,j} \mathbf{n}_{i,j}^3\}_{1 \leq i < j \leq N}$. Then equation (30) with $* = 3$ remains valid, where $p_{ij}^{3,\mathfrak{h}} = p_{N,\beta}^{3,\mathfrak{h}}$ if i, j are either both in Λ_{aa} or both in Λ_{ad} or both in Λ_{da} or both in Λ_{dd} and $p_{ij}^{3,\mathfrak{h}} = 0$ otherwise. The asymptotic behavior as $N \rightarrow \infty$ of $p_{N,\beta}^{3,\mathfrak{h}}$ is*

$$p_{N,\beta}^{3,\mathfrak{h}} = \frac{4\beta^2}{N} (1 + O(\frac{1}{\sqrt{N}})). \quad (38)$$

Proof. As in the proof of Lemma 4.2, we apply Lemma 4.1 in the guise of (34) with $\varepsilon_{ij} = \sigma_i^1 \sigma_j^1$ again, but this time with $k = 3$. Now $\mathbf{b}_{ij} \mathbf{n}_{ij}^3 = \mathbf{b}_{ij}^{12} \mathbf{n}_{ij}^3 = 0$ unless i, j are either both in Λ_{aa} or both in Λ_{ad} or both in Λ_{da} or both in Λ_{dd} , in which case we have $\sigma_i^1 \sigma_j^1 = \sigma_i^2 \sigma_j^2 = \sigma_i^3 \sigma_j^3$, $\gamma_{ij}^{3,+} = 6\beta/\sqrt{N}$, $\gamma_{ij}^{3,-} = 0$, and $\mathbf{b}_{ij}^{12} \mathbf{n}_{ij}^3$ is an increasing function of $(K', -U)_N^3$. The remainder of the proof, which closely mimics that of Lemma 4.2, is straightforward.

Proof of Theorem 1. We denote by $D_{\alpha,j}^{\text{CMR}} = D_{\alpha,j}^{\text{CMR}}(N, \beta)$ for $\alpha = a$ or d the density (fraction of N) of the j 'th largest CMR cluster in Λ_α and by $S_j^{\text{RG}} = S_j^{\text{RG}}(N, p_N)$ the number of sites in the largest cluster of the random graph $G(N, p_N)$. Recall that $D_\alpha = D_\alpha(N, \beta)$ denotes the density (fraction of N) of Λ_α and that $\max\{D_a, D_d\} = [1 + |Q|]/2$, $\min\{D_a, D_d\} = [1 - |Q|]/2$. Then Lemma 4.2 implies that, conditional on σ^1, σ^2 ,

$$N D_{\alpha,1}^{\text{CMR}}(N, \beta) \gg S_1^{\text{RG}}(N D_\alpha(N, \beta), (2\beta c_\rho/\sqrt{N}) + O(1/N)) \quad (39)$$

with $c_\rho > 0$. By separating the cases of small and not so small $D_\alpha(N, \beta)$ and using the above stochastic domination combined with the facts that $m^{-1}S_1^{\text{RG}}(m, p_m) \rightarrow \theta(1) = 1$ (in probability) when $m, mp_m \rightarrow \infty$, and that $D_{\alpha,1}^{\text{CMR}} \leq D_\alpha$, it directly follows that $D_\alpha - D_{\alpha,1}^{\text{CMR}} \rightarrow 0$ and hence also that $D_{\alpha,2}^{\text{CMR}} \rightarrow 0$ (in probability). This then directly yields (3), (4) and (5) and completes the proof of Theorem 1.

Proof of Theorem 2. The proof mimics that of Theorem 1 except that (39) is replaced by

$$S_1^{\text{RG}}(N D_\alpha, \frac{2\beta^2}{N} + O(N^{-3/2})) \gg N D_{\alpha,1}^{\text{TRFK}} \gg S_1^{\text{RG}}(N D_\alpha, \frac{2\beta^2}{N} + O(N^{-3/2})), \quad (40)$$

where the two terms correcting $2\beta^2/N$, while different, are both of order $O(N^{-3/2})$. We then use the fact that $m^{-1}S_1^{\text{RG}}(m, p_m) - \theta(mp_m) \rightarrow 0$ when $m \rightarrow \infty$ to conclude that $D_{\alpha,1}^{\text{TRFK}}(N, \beta) - \theta(2\beta^2 D_\alpha(N, \beta)) D_\alpha(N, \beta) \rightarrow 0$ which yields (7) and (8). To obtain (9), we need to show that $D_{\alpha,2}^{\text{TRFK}} \rightarrow 0$. But this follows from what we have already showed, from the analogue of (40) with $D_{\alpha,1}^{\text{TRFK}}$ replaced by $D_{\alpha,1}^{\text{TRFK}} + D_{\alpha,2}^{\text{TRFK}}$ and S_1^{RG} replaced by $S_1^{\text{RG}} + S_2^{\text{RG}}$, and by the fact that $m^{-1}S_2^{\text{RG}}(m, p_m) \rightarrow 0$ in probability when $m \rightarrow \infty$. This completes the proof of Theorem 2.

Proof of Theorem 4. Theorem 1 implies that any limit in distribution of $(D_1^{\text{CMR}}, D_2^{\text{CMR}}, D_3^{\text{CMR}})$ coincides with some limit in distribution of $(\frac{1+|Q|}{2}, \frac{1-|Q|}{2}, 0)$. The claims of the theorem for $0 < \beta \leq 1$ then follow from Theorem 3 and for $1 < \beta < \infty$ from that theorem and Properties P1' and P2'.

Proof of Theorem 5. Theorem 2 implies that any limit in distribution of $(D_1^{\text{TRFK}}, D_2^{\text{TRFK}}, D_3^{\text{TRFK}})$ coincides with some limit in distribution of $(\theta(\beta^2(1 + |Q|))(\frac{1+|Q|}{2}), \theta(\beta^2(1 - |Q|))(\frac{1-|Q|}{2}), 0)$. The claims of the theorem for $0 < \beta \leq 1$ then follow from Theorem 3 and the fact that $\theta(c) = 0$ for $c \leq 1$ while the claims for $1 < \beta < \infty$ follow from Theorem 3 and Properties P1' and P3'. Note that P3' is relevant because $\theta(\beta^2(1 - |Q|)) > 0$ if and only if $\beta^2(1 - |Q|) > 1$ or equivalently $|Q| > 1 - (1/\beta^2)$.

Proof of Theorem 6. This theorem is an immediate consequence of Theorem 1 (which can be applied to CMR percolation using any pair of replicas ℓ, m) since $D_1^{\ell m} - D_2^{\ell m} = |Q^{\ell m}|$.

Proof of Theorem 7. The identity $Q^{12} = D_a - D_d$ which involves only two out of three replicas may be rewritten in terms of three-replica densities as

$$Q^{12} = D_{aa} + D_{ad} - D_{da} - D_{dd}. \quad (41)$$

The corresponding formulas for the other overlaps are

$$Q^{13} = D_{aa} - D_{ad} + D_{da} - D_{dd}, \quad (42)$$

$$Q^{23} = D_{aa} - D_{ad} - D_{da} + D_{dd}. \quad (43)$$

Combining these three equations with the identity $D_{aa} + D_{ad} + D_{da} + D_{dd} = 1$, one may solve for the $D_{\alpha\alpha'}$'s to obtain

$$D_{aa} = (Q^{12} + Q^{13} + Q^{23} + 1)/4, \quad (44)$$

$$D_{ad} = (Q^{12} - Q^{13} - Q^{23} + 1)/4, \quad (45)$$

$$D_{da} = (-Q^{12} + Q^{13} - Q^{23} + 1)/4, \quad (46)$$

$$D_{dd} = (-Q^{12} - Q^{13} + Q^{23} + 1)/4. \quad (47)$$

In the equilateral triangle case where $|Q^{12}| = |Q^{13}| = |Q^{23}|$ and also $Q^{12}Q^{13}Q^{23} > 0$ (from Property P5'), one sees that the corresponding values of the $D_{\alpha,\alpha'}$'s lie in $\mathbf{R}_{\text{ultra}}^{4,+}$ with $x_{(1)} > x_{(2)} = x_{(3)} = x_{(4)}$. In the isosceles triangle case (with $Q^{12}Q^{13}Q^{23} > 0$), one is again in $\mathbf{R}_{\text{ultra}}^{4,+}$, but this time with $x_{(1)} > x_{(2)} > x_{(3)} = x_{(4)}$.

Proof of Theorem 8. Here we use Lemma 4.3 to study the densities (fractions of N) $D_{\alpha\alpha',j}^3$ (where α and α' are a or d) of the j 'th largest cluster in $\Lambda_{\alpha\alpha'}$ formed by the bonds $\{ij\}$ with $\mathbf{b}_{ij}\mathbf{n}_{ij}^3 = 1$. Similarly to the proof of Theorem 5, we have

$$S_1^{\text{RG}}(N D_{\alpha\alpha'}, \frac{4\beta^2}{N} + O(N^{-3/2})) \gg N D_{\alpha\alpha',1}^3 \gg S_1^{\text{RG}}(N D_{\alpha\alpha'}, \frac{4\beta^2}{N} + O(N^{-3/2})) \quad (48)$$

and use this to conclude that $D_{\alpha\alpha',1}^3 - \theta(4\beta^2 D_{\alpha\alpha'}) D_{\alpha\alpha'} \rightarrow 0$ and $D_{\alpha\alpha',2}^3 \rightarrow 0$. Noting that $D\theta(4\beta^2 D)$ is an increasing function of D , the rest of the proof follows as for Theorem 7.

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